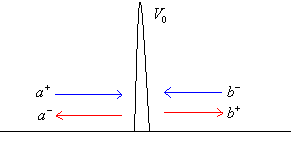
**Scattering Examples**

Doing some more examples, this time with multiple scatterers. Well, before I can do that, I have to redo the S and M matrices for a single δ potential, generalizing this time to it being at an arbitrary position:

**Example. 1D δ potential at arbitrary location x1 (Scattering Matrix)**

Now let’s work out the scattering matrix. Consider a δ function potential V(x) = V0δ(x-x1). What is the scattering matrix for such a potential?



Well on either side we have:



Continuity at the boundary requires,



and the equation’s differentiability implies…from the QM folder,



And now we want to separate the a+ and b- terms (and adding an identical term to the RHS of eq. 2, and then dividing by 2 to compensate…



and,



Continuing,



And now inverting…



and now we’re gonna simplify the matrix…



and still going,



So we have:



This reduces to the origin S when x1 = 0, so that’s good. Is this unitary? Well S-1 is:



and Sⴕ is given by:



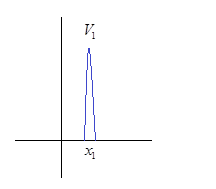
So these match! Finally, we worked out in the general S-matrix file, that S changed as follows under a translation:



and we see this so.

**Example. 1D δ potential at arbitrary location x1 (Transfer Matrix)**

Suppose we have:



Wavefunction is:



Continuity requires:



And the delta-function requires a discontinuity in the wavefunction. The discontinuity is:



So we have:



So these are our 4 equations. We want to solve for C, D in terms of A, B.



and,



We can express this in matrix form as:



So we have:



This matches our origin result when x1 = 0. So good. So our transfer matrix is:



Do both M and S match the general form?



Comparing,



to the general form of S, we see that the matrix elements of M should be:



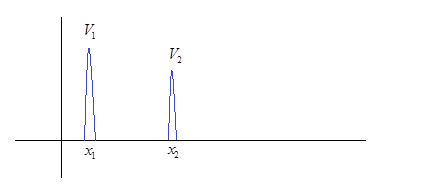
and this is so! Finally, we worked out in the general M-matrix file, that M changes as follows, under a general translation:



and we see this so.

**Example. Two δ function Potentials**

Suppose we have two 1D with different strengths? What is M for them?



Wavefunction is:



Continuity requires:



And the delta-function requires a discontinuity in the wavefunction. The discontinuity is:



So we have:



So these are our 4 equations. We want to solve for E, F in terms of A, B. So I will solve the C, D equations and then plug them into the E, F equations:



We can express this in matrix form as:



And it is clear from the equations that the relationship of EF to CD will be the same, with the replacement of V1 → V2 and x1 → x2. And so,



Now the transfer matrix actually relates current amplitudes on right to those on left. So really we need relationship between (E/√(k/m), F/(√k/m)) and (A/√(k/m) B/√(k/m)). But of course, since k factors are same for each coefficient, we get the same result. So then, our compound M is:



etc. And so in general, if we had N such potentials, then the transfer matrix would be:



and knowing M, we could then extract t, r, etc., to get T and R.

**Checking symmetry**

Does this matrix obey the symmetry properties?



The first requires:



So this checks out. And the second requirement is:



**Looking for Bound States**

Let’s go back to M for the two potentials:



And let’s make the situation symmetric so V1 = V2 = V, and x1 = -a/2, x2 = a/2.



M is anti-symmetric and off-diagonals are imaginary. Recalling,



and that for TRS, t´ = t, we have:



and for r we have:



by parity symmetry, r´ = r too. Let’s look for positive parity eigenstates. This would be at poles of t + r,



Working this into the phase form is not cool. So let’s just skip to the analysis. We know that the poles of S correspond to bound state states or resonant states. Let’s consider taking V = ∞. Then this is effectively an infinite box problem. And indeed we find poles of S++ at 2ka = 2nπ → k = nπ/a. Now we know from our analysis in the Eigenfunctions folder that there are actual bound states at finite (negative) V. So there must be a pole yet, which should show up on the imaginary axis. The equations would be:



Compared to our Eigenfunction file result:



Must be the same after some algebra…Let’s look for resonant states, such as the aforementioned hard-wall eigenstates. To that end, let’s look for complex zeros near kR = nπ/a, backing off from the V = ∞ limit. We should expect roots of the form,



So making this expansion…



Setting orders to zero,



and



So we find an approximate pole at:



At least I think so. Anyway, so we see that the lifetime only comes in at second order in the perturbation. And the pole is in the lower-half plane. This makes sense, as then E = k2/2m ~ Ereal – iEimag, and the evolution will be exp(-iEt) ~ exp(-iErealt – E­imagt). So we get exponential decay as needed.

**Example. Two scatters**

A 1D scattering potential is described by the S-matrix,



What is the transmission coefficient? What is M? Well since,



we see that:



And since M is given by:



it follows that M is:



And of course we also see that the transmission coefficient is:



Now place another identical scatterer a distance Δx = λ/4 away (λ is wavelength of electron = 2π/k). What will be the net M and T? The additional scatterer is described by:



So the net transfer matrix is:



Can then extract the transmission and reflection amplitudes,



and so,



What if we had moved the scatterer to a distance Δx = λ/2 away? Then?



So it’s the same as M(0). Now let’s do,



So now,



Let’s look at a general formula. And compare. So we could say:



We could get t = t´ (in this case, because of parity symmetry) via:



Note we also use r = r´ from Parity symmetry. Say we place 3 such scatters side by side, a distance Δx = λ/2 apart. Then we have:



Clearly, if we place n such scatters together, we’ll get:



What is the transmission coefficient?

